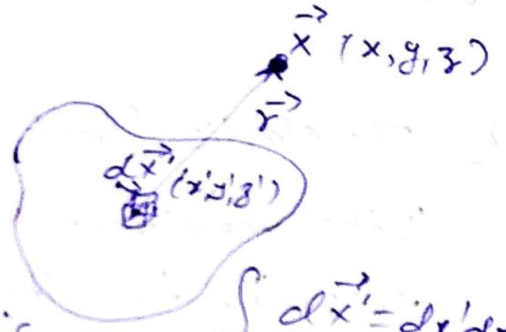
Divergence of magnetic field $\vec{B}$:

Magnetic field $\vec{B}$ at the point $\vec{x} = (x, y, z)$ in terms of volume current density $\vec{J}(\vec{x})$ is given by the Biot-Savart law



$d\vec{x}' = dx'dy'dz'$

$\vec{x} = (x, y, z)$

$\vec{x}' = (x', y', z')$

$\vec{x} - \vec{x}' = \vec{r}$

$|\vec{x} - \vec{x}'| = |\vec{r}| = r$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}') \times \hat{r}}{r^2}\, d\vec{x}' \qquad \text{--- (1)}$$

Next, we take divergence of both sides of equation (1)

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left[\vec{J} \times \frac{\hat{r}}{r^2}\right] d\vec{x}' \qquad \text{--- (2)}$$

In the above expression on the right-hand side, we use the following identity of product rules-

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot \left[\vec{J} \times \frac{\hat{r}}{r^2}\right] = \frac{\hat{r}}{r^2} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\hat{r}}{r^2}\right)$$

In the above expression $\vec{J} \equiv \vec{J}(\vec{x}')$ and

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z},$$

therefore $\nabla \times \vec{J} = 0$

Thus we obtain $\nabla \cdot \left[\vec{J} \times \frac{\hat{r}}{r^2}\right] = -\vec{J} \cdot \left[\nabla \times \frac{\hat{r}}{r^2}\right]$ — (3)

Next, we need to obtain $\nabla \times \frac{\hat{r}}{r^2}$

This can be calculated using spherical coordinates (H.W.) as well as Cartesian coordinate system

If we shift $\vec{x}'$ to the origin, $\vec{x} = \vec{r}$.

$$\nabla \times \frac{\hat{r}}{r^2} = \nabla \times \frac{\vec{r}}{r^3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad \text{— (4)}$$

Let $\frac{\vec{r}}{r^3} = \vec{A}(x, y, z)$
$= \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$

where $A_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}$, $A_y = \frac{y}{(x^2+y^2+z^2)^{3/2}}$, $A_z = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

Let us take $\hat{i}$ component (Eq. 4) $\Rightarrow \hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]$

$$= \hat{i}\left[\frac{\partial}{\partial y}\frac{z}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \hat{i}\left[z\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2} - y\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\frac{3\hat{i}}{2}\left[z(x^2+y^2+z^2)^{-5/2}(2y) - y(x^2+y^2+z^2)^{-5/2}(2z)\right]$$

$$= -3\hat{i}\left[yz(x^2+y^2+z^2)^{-5/2} - yz(x^2+y^2+z^2)^{-5/2}\right] = 0$$

Similarly for $\hat{j}$ and $\hat{k}$ components we obtain zero (H.W.)

Thus we obtain $\nabla \cdot \left[\vec{J} \times \frac{\hat{r}}{r^2}\right] = 0$ — (5) (From eq^n 3)

using eq^n (2) and (5), we obtain

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$\Rightarrow$ Divergence of magnetic field is zero.